**Chaotic Evolution and Strange Attractors**
**David Ruelle**
**Lezioni Lincee**
**1989**

## Part I. Steps to a deterministic interpretation of chaotic signals.

### 1. Descriptions of Turbulence.

How does one explain a situation in which one gets a signal (time series) which is chaotic?

One way is with *stochastic evolution equations* of the form: $\frac{d\,\boldsymbol{x}(t)}{dt} = F(\boldsymbol{x}(t)) + w(t)$ where w(t) is the noise term (eg a stochastic process)

Another way involves the presence of many oscillators in a *quasi periodic attractor:* $\boldsymbol{x}(t) = f(w_1 t, w_2 t, \ldots, w_k t) = f(\phi_1, \phi_2, \ldots, \phi_k)$
The number k of frequencies in a quasiperiodic motion id defined as the minimum number of rationally independent frequencies of the form which are present in the Fourier transform (ie the modes of the system)

An indicator of the qualitative nature of motion is the power spectrum or frequency spectrum which can distinguish a periodic spectrum, a quasiperiodic spectrum and a continuous spectrum.

The main problem with the quasiperiodic theory of turbulence is that when there are nonlinear coupling between oscillators, it very often happens that the time evolution does not remain quasiperiodic. This feature is called sensitive dependence on initial conditions and turns out to be the conceptual key to reformulating the problem of turbulence.

This type of motion, although purely deterministic, has those stochastic features referred to as chaos.

The existence of sensitive dependence on initial conditions was initially noticed by Hadamard in the late 1800's in studies of geodesic flow on surfaces with constant negative curvature. While it was remembered by mathematicians, it was forgotten by physicists until Lorenz in 1963.

$$\frac{dx}{dt} = -\sigma x + \sigma y$$
$$\frac{dy}{dt} = -xy + rx - y$$
$$\frac{dz}{dt} = xy - bz$$

The Lorenz system describes a set of points on a trajectory of a geometrical object known as a strange attractor. A strange attractor is an infinite set of points in a m-dimensional space, which represents the asymptotic behaviour of a chaotic system.

A time evolution which is chaotic in this sense usually exhibits as continuous power spectrum.

The power spectrum is only an indicator but not very useful for specific analysis because the 'dimension' of chaotic motion is not related to the number of independent frequencies.

### 2. A bit more on turbulence. The Navier-Stokes equation.

$$\frac{dv_i}{dt} + \sum_j v_j \frac{dv_i}{dx_j} = \nu \Delta v_i - \frac{1}{\rho} + f_i$$
$$\sum_{i=1}^{d} \frac{dv_i}{dx_i} = 0$$

where, the second equation is the incompressibility condition, f is the external force per unit volume, ρ is the constant density, p is the pressure and v is the constant kinematic viscosity.

These equations have a unique solution in 2 dimensions but in 3 dimensions it is not known if the velocity develops singularities. The existence of singularities have been studied and isolated to regions in space-time. However, the whole problem may be non-physical and simply not occur in real fluids where, for instance, one has to avoid large negative pressures (cavitation) – a condition which is ignored in the mathematical models.

$m^\alpha$ is called the Hausdorff measure in dimension α. Given a non-empty set S, with a metric, and r>0, denote by σ a covering of S by a countable family of subsets $\sigma_k$ of diameter $r_k \leq r$

$$m^\alpha(S) = \lim_{r \to 0_+} m_r^\alpha(S),$$
*where*
$$m_r^\alpha = \inf_\alpha \left\{ \sum_{k=1}^{\infty} (r_k)^\alpha \right\}$$

When r decreases, the infimum extends over smaller and smaller classes of coverings and hence $m^\alpha$ increases, or at least does not decrease.

Generally, a system has a stable equibrium state represented by a fixed point attractor in the phase space M. The aim of the geometric theory of chaos is to give some predictions of the following form: if the attractor undergoes some qualitative changes as an experimental control parameter μ is varied, then certain other changes are likely to happen as μ is varied further. We are far from a complete classification of possible scenarios, but three are relatively well understood:

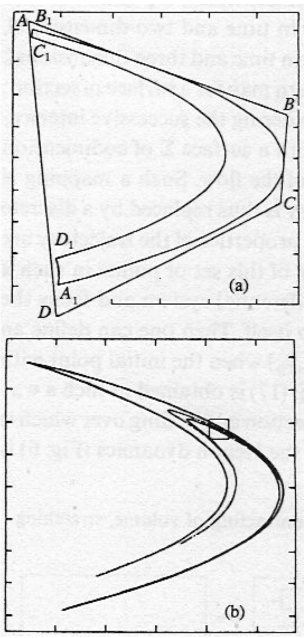
1. Ruelle-Takens scenario through quasiperiodicity;
2. Feigenbaum scenario through period doubling;
3. Pomeau-Manneville scenario through intermittency.

Turbulence remains a very difficult topic but we do know that chaos is involved and that any theory omitting chaos is inadequate (eg those that start with stochastic equations or consider purely quasiperiodic motions).

Deterministic noise cannot be removed by improving experimental apparatus.

### 3. The Hénon mapping.

$$x(t+1) = y(t)+1-a[a(t)]^2$$
$$y(t+1) = bx(t)$$




The Hénon model is more handy for numerical explorations because it is discrete in time and 2 dimensional (in contrast to Navier-Stokes).

It simulates the stretching in one direction and folding over which a typical behaviour with the Lorenz system.

### 4. Capacity and Hausdorf Dimension.

Roughly speaking, the dimension of a set is the amount of information needed to specify points in it accurately. The Lebesgue measure considers the combined length of disjoint segments. Often we are interested in invariant sets with Lebesgue measure zero (e.g. the Hénon map) and in which is bigger.

The *capacity* is given by:

$$dim_K(A)=\lim_{r\to 0}\sup\frac{logN(r,a)}{\log(1/r)}$$

The Hausdorff dimension of A is defined by considering the behaviour of $m^\alpha(A)$ not as a function of A, but as a function of α.

$$dim_H(A) = \sup\{\alpha : m^\alpha(A)=+\infty\} = \inf\{\alpha : m^\alpha(A)=0\}$$

For the Cantor set the capacity and the Hausdorf dimension coincide (log 2 / log 3).

Mandelbrot introduced 'fractal dimension' to refer to the Hausdorf dimension but other authors use the same term for the capacity.

The information dimension $dim_H\rho$ is defined as the minimum of the Hausdorf dimension of the sets A for which ρ(A)=1, where ρ is a probability measure.

### 5 Attracting Sets and Attractors.

The Hénon mapping is a dissipative system because each mapping contracts by a factor b (the Jacobian).

- The Lorenz system is continuous and $R^3$
- The Hénon system is discrete and $R^2$
- The Navier-Stokes system is continuous and infinite
- Hadamard's geodesic flow is continuous on a compact manifold.

We introduce the nonlinear time evolution operators $f^t$, with the property $f^0=x$ and $f^{t_1+t_2}x=f^{t_1}\circ f^{t_2}x$

Set A will be called an attracting set with fundamental neighbourhood U if it satisfies:

1. Attractivity: for every open set $V\supset A$ we have $f^t U\subset V$ for all sufficiently large t.
2. Invariance: $f^t A=A$ for all t

The attracting Sets basin of attraction is defined to be the set of initial points x such that $f^t x$ approaches A.

The operational definition of an attractor is the set on which experimental points $f^t x$ accumulate for large t. It should be invariant but need not be stable under noise.

3. Irreducibility: one can choose a point x'ϵA such

that for each x∊A there is a positive t such that $f^t x'$ is arbitrarily close to x (topological transitivity)

4. Sensitive dependence on initial conditions: A strange attractor is defined by the fact that its asymptotic measure has a positive characteristic exponent.

An attractor (eg Feigenbaum attractor) can have a fractal structure without being chaotic; an attractor which is not an attracting set.

While sensitivity to initial conditions means that a modest number of iterations may have results at “opposite ends” of an attracting set, the statistical properties of the results are invariant.

5. Stability under small random perturbations: small random perturbations must be asymptotically concentrated on attractors and the asymptotic measure ρ must be stable under such perturbation.

**6 Extracting geometric information from a time series.**

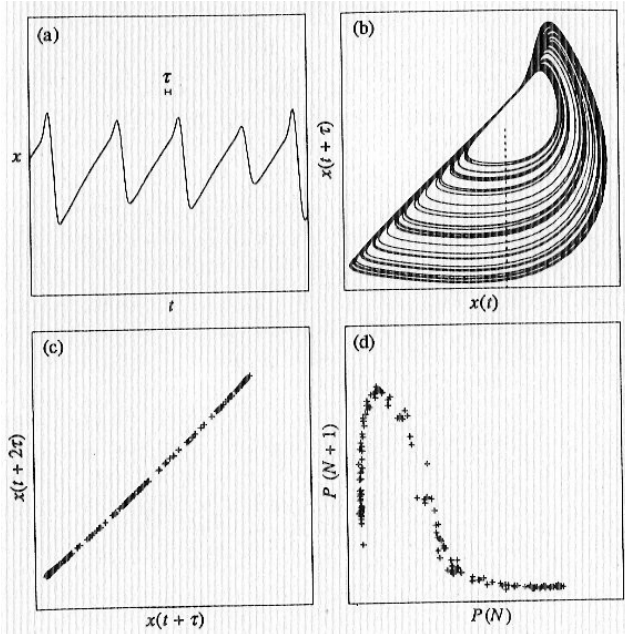
Phase space pictures can be reconstructed from the experimental observation of a single coordinate x(t), bypassing the detailed knowledge of the underlying dynamics.

The general technique of this approach is to generate several different scalar signals $x_k(t)$ from the original x(t) in such a way as to reconstruct an N-dimensional space where, under some conditions, we can obtain a good representation of the attractor.

The easiest way to do this it to use time delays (originally derivatives were used: x1=x(t), x2=ẋ(t) ...)

$$x_k(t) = x(t+(k-1)\tau) \quad with \quad k=1 \ldots N$$

There are some theorems which state that, in order to obtain a good projection of πA (without trajectories crossing each other), N must be about twice the Hausdorff dimension of A.




(a) The time series x(t) of the bromide concentration in the Belousov-Zhabotinsky reaction.
(b) Plot of the reconstructed attractor in the plane (x(t), x(t+τ)).
(c) Poincare section of the attractor along the cut.
(d) First return map of the Poincare section.

This procedure is only feasible with low-dimensional attractors.

## Part II The ergodic theory of chaos

**7 Invariant probability measures.**

The essential fact is the existence of a probability measure ρ on M, which is invariant under the time evolution of $f^t$.

The notion of Radon measure is that one can *define* the measure ρ by:

$$\rho(\phi) = \int \phi(x)\rho(dx)$$

We say that any continuous linear functions in $C^0(M)$ - the class of continuous functions on a compact metric space M - is a measure on M (Radon measure)

invariance :
$$\rho(\phi) = \rho(\phi \circ f^t)$$
$$where \, (\phi \circ f^t)x = \phi(f^t)$$

Consider the map:

$$f(x) = \begin{cases} 2x, & \text{if } x \in [0, \frac{1}{2}) \\ 2x - 1, & \text{if } x \in [\frac{1}{2}, 1) \end{cases}$$

This mapping can be considered to be on the unit circle (in the sense that it wraps around onto it own perimeter) or as a shift operator (in the sense that the most significant decimal is removed and all the less significant digits are promoted). Clearly ρ is invariant under the shift.

Ergodicity: An invarient probability measure ρ is ergodic or indecomposable if it does not have a nontrivial convex decomposition.

$$\rho = \alpha \rho_1 + (1 - \alpha) \rho_{2,} \; with \; \alpha \; ne$$