Mathematics Today
Lynn Arthur Steen

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Most technologies rely on mathematics

Even most educated people people have mathophobia and just think about arithmetic.

The communication about math is frustrated by the jargon and the practitioners

Mathematics has a 2000 year history but is still active.

19th century: refined calculus and opened up infinite sets and non-euclidean geometries. Both were rooted in Fourier's insight.

20th century: Hilbert's 23 unsolved problems were devastated by Godel's uncertainty proof.

Algebra analysis and topology are the three core disciplines.

Pure mathematics sits at the core as support for the applied mathematical sciences.

It is difficult to predict how pure mathematics will be applied and often slow to develop new pure mathematics to support a particular application.

Beauty and elegance have much to do with the value of a mathematical idea and there is remarkable agreement as to what is important and what dull.

Mathematics today is healthy, vibrant and useful but sits on a fault lines of societies intellectual landscape. This mild hostility causes mathematicians to retreat into their own with the danger of loosing support.

Mathematics – Our Invisible Culture
Allen L. Hammond

There are as many mathematicians in USA as there are physicists and economists and mathematics have never been more active but it is invisible.

Is mathematics essentially remote, or poorly communicated or is ignorance to blame?

An interview with 3 mathematician: Bers (analysis), Sullivan (topology) and Puckette (student numerics)
Sullivan pg 18 “In one period I would work like this. I would wake up in the morning thinking about something, and think about it until I ran into some other talking mathematician, then I would start talking about it. I would talk about it all day long, go through all kinds of deformations and interactions about what I was doing. I would go home in the evening to my family, have supper, do what I had to do to lead a normal life, then sit on the couch and start thinking about it again. I would think about it until I fell asleep, then start over again the next day. .... You have to have an overall prejudice or goal.”

Research is both discovery and invention.

Mathematics is the most intellectual of the arts with profound links to reality and there is an ultimate decidable truth.

Simplicity and beauty is important.

Mathematics is built on concepts on numbers and space.

Young uneducated children can be taught advance algebra by the right teacher but the mathematical language is obscure.

Mathematics requires enormous effort which most usually fails.

Mathematics is done for the sparse moments of understanding.

Sullivan pg 29 “Its harder to understand mathematics than it is to do it ..... [but] it's hard to find something really good...... There's this dull pain all the time you have to think and concentrate and try to understand. But there are more frequent levels of understanding – a sort of superficial understanding when you can just say something empirically, then a little better understanding when you start seeing relationships, then [the best level] when you find a totally new area, which happens very rarely.”

Mathematicians must be honest because there is only one truth.

Only in mathematics can a 20 year old lecture to masters

Every art is founded on the study of pattern and Mathematics is the most powerful technique for understanding pattern.

A mathematical idea is permanent over millennium but often only accessible to a few.

Number Theory
Ian Richards

Number theory is concerned with the properties of whole numbers and is as old as geometry

We can think about numbers much larger than any computer can handle.

Mathematics mainly involves intuition and the ability to grasp theoretical ideas.

Gauss was exceptional at everything!

Diophantine equations are concerned with the whole number solution of equations.

Every positive whole number is the sum of four or fewer perfect squares is easy to state but (very) difficult to prove.

Euler was exceptional, devout, prolific & humble.

Lagrange was exceptional, sceptical, not prolific but influential none the less.

The Riemann hypothesis asserts that the zeros of the Riemann zeta-function all lie on a single line but it remains unproven. $1-\frac{1}{2^s}+\frac{1}{3^s}-\frac{1}{4^s}+-...=0$

Prime numbers form the 'atoms' from which all other numbers are built by multiplication.

There are many unproven Prime conjectures.

Twin primes are pairs of consecutive odd numbers, both of which are prime. That they are infinite in number is unproven.

The sieve of Erathosthenes operates by removing non primes from a range (ie multiples of 2 then 3 then 5 etc.) and each subsequent step depends on the result of the last.

Brun's principle of inclusion-exclusion is a remarkable simple method of estimating the number of primes in a range without actually knowing what they are.

The prime number theorem states that the number of primes less that x is approx $^{x}/_{\log x}$ (Legendry and Gauss). Further more, the approximation improves as x approaches ∞. The proof turned out to be very lengthy and hence impossible to appreciate.

A proof of the Riemann hypothesis underpins a number of important results in prime number theory.

It has been proven that at least 1/3 of the solutions remain on a straight line.

Fermats last theorem asserts that for integers x,y,z, none of which are zero, and an integer n>2, $x^n+y^n=z^n$ but it has not been reproved.

Equations in two variables x,y of degree three or more have only a finite number of solutions.
There are only 9 Gauss numbers 1, 2, 3, 7, 11, 19, 43, 67, 163.

There are only two powers of Integers which differ by exactly one $2^3=8$ and $3^3=9$.

The Chinese Remainder Theorem explores what happens when two periodic events are combined. It enables a machine to give more accuracy that it is normally designed for.

The general form of odd/ even is congruence modulo x.

Groups and Symmetry
Jonathon Alperin

The importance of abstractions is nowhere more apparent than in the concept of a group.

The symmetry of an equilateral triangle is a way of moving the triangle so as not to disturb its appearance; a symmetry thus preserves all the relations of distance and angle between the vertices and edges of the triangle.

A group is a collection of objects and a rule for multiplying any two of them which has three properties:

There is an object e in the group such that xe = x = ex for any object x in the group.

For any object x there is an object y in the group with xy = e = yx.

Whenever x, y, and z are in the group then (xy)z = x(yz).

In general, if we look at an equilateral n-gon, that is, a polygon with n sides, then there will be a symmetry group consisting of 2n operations.

Three of the six operations on the equilateral triangle do not turn over the triangle. These three operations by themselves also form a group.

Another important example of a group is the rearrangement of a set (permutations). Exactly half of the permutations keep the numbers in order. A permutation is called even if the numbers of such pairs which are not kept in order is even (and odd otherwise).

Galois devised groups that reflect the symmetry properties of the roots of polynomial equations. This is one of the most remarkable ideas in the history of mathematics that enables the solution to a polynomial equation to be expressed by the elementary arithmetic operations and the extraction of roots exactly when the Galois group has has certain special properties.

The essential feature of Lie groups is continuity. They are described by continuously varying parameters like the angle in the rotation of a line or a plane about an origin. There are very fundamental connections between the continuous Lie groups and the finite and discrete groups.

Groups are highly flexible and easy to define as it only requires the definition of multiplication.

The insights gained in one group can often be transferred to others.

In group theory the fundamental objects are the so called simple groups (but they are not simple to understand or find). Simple groups are those that cannot be decomposed into smaller sub groups where the subgroups themselves have a well behaved definition of multiplication between them.

Lie groups have been completely classified

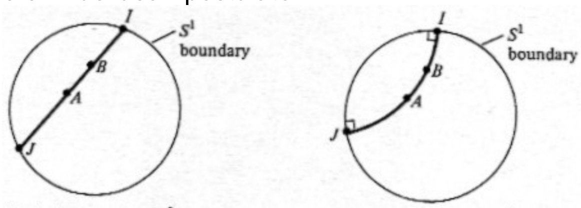
Classification of algebraic groups also results in groups of the Lie type.

There are 18 families of simple finite groups including 16 Lie, cyclic and alternating groups.

The Geometery of the Universe
Roger Penrose

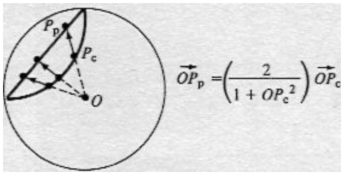
Differential geometry is more subtle and flexible than the Euclidean beginnings.

Lobachevskian (or hyperbolic) geometry abandons the 5th Euclidean postulate.



Projective model of L^2 Conformal model of L^2

The Lobachevskian distance AB=$\epsilon \log_e$(IA.JB/IB.JA) where ϵ=½ for the projective model and ϵ=1 for the conformal model. The boundary circle S^1 represents infinity for L^2.



If the two diagrams are superimposed, then the point Pp of the projective model is obtained from the corresponding point Pc of the conformal model by moving it radially outward from the centre.

Another type of 2D geometry is obtained by representing a spherical surface S^2 on E^3. The 'straight lines' are the 'great circles' and distance is the arc length. Note that the 'straight lines' have the same topology as circles and intersect one another in pairs rather than points.

The elliptical space P^2 is derived from S^2 by identifying antipodal points such that each point of P^2 corresponds to 2 points in S^2 which lie on a straight line thru its centre.

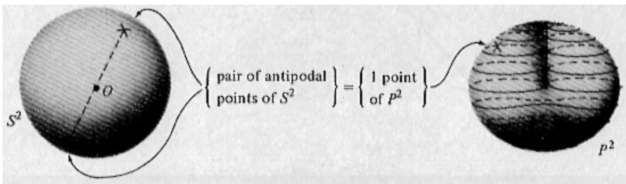


Figure 5. Spherical and elliptic geometry.

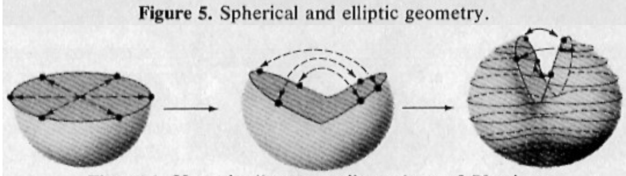


Figure 6. How the "cross-cap" topology of P^2 arises.

The topology of P^2 can be described as a sphere with a cross cap.

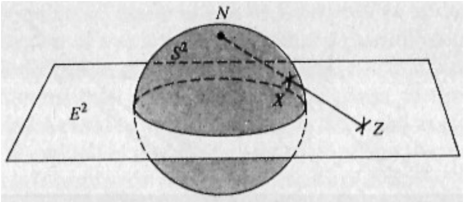


Figure 7. Stereographic projection of S^2.

S^2 can be represented on the plane by stereographic projection with the plane passing thru the equator. This is a conformal representation in the sense that angles are maintained.

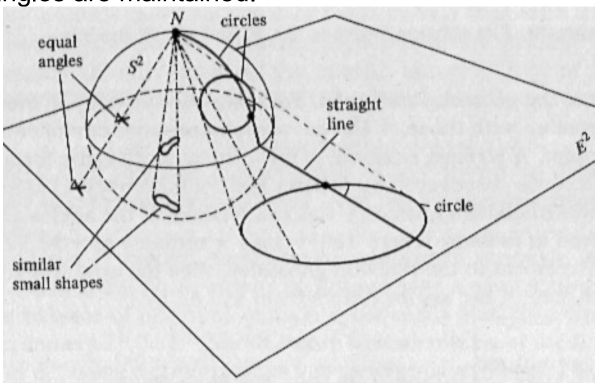


Figure 8. Effects of the stereographic projection.

P^2 can be similarly represented by dropping the plan to be tangential to the 'south pole'. This representation is projective as great circles on S^2 are carried to straight lines on E^2.

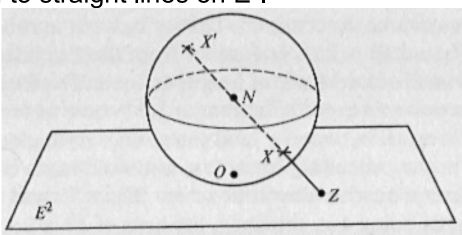


Figure 9. Projection of P^2 to E^2.

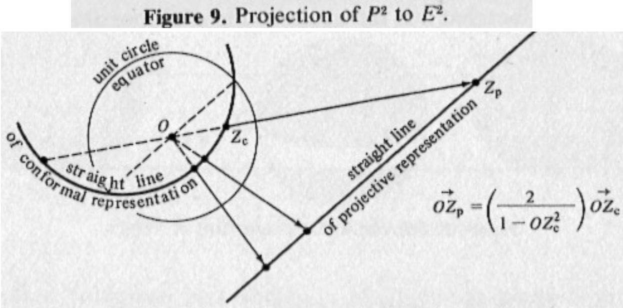


Figure 10. The relation between the S^2 and the P^2 models.

If the angles α, β, γ are measured in radians and the sides are straight then the area Δ is given by:

$$\Delta = \left\{ \begin{matrix} \pi - (\alpha + \beta + \gamma) & for\ L^3 \\ (\alpha + \beta + \gamma) - \pi & for\ S^2 \text{ and } P^2 \end{matrix} \right\}$$

These geometries represent the simplest deviations from Euclidean geometry and are of special interest as the best candidates for the large scale structure of the universe. However the radius of curvature of the spacial universe is extremely large (10^{10} light years) so the geometry appears Euclidean on our infinitesimal scale.

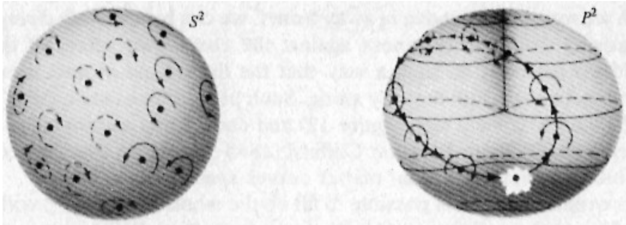


Figure 11. Orientability of S^2 and nonorientability of P^2.

By travelling around certain closed paths on P^2 we can return to the starting point but with the sense of handedness reversed. However this is not the case in any odd number of dimensions.

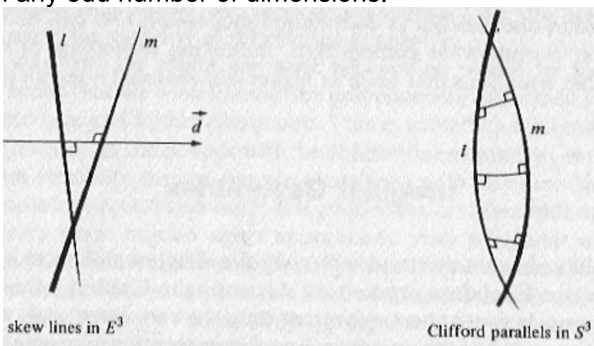


Figure 12. Clifford parallels.

The geometries L^2 and P^2 (or S^2) differ from Euclid with regard to the concept of parallel lines. In L^2 two initially parallel lines diverge while the opposite is true in P^2 and S^2. This corresponds for 3D but a slightly different concept of parallel can be introduced for P^3 and S^3. Clifford parallels introduce a skew between lines l & m (fig 12) that balances the converging effect of the non-Euclidean geometry.

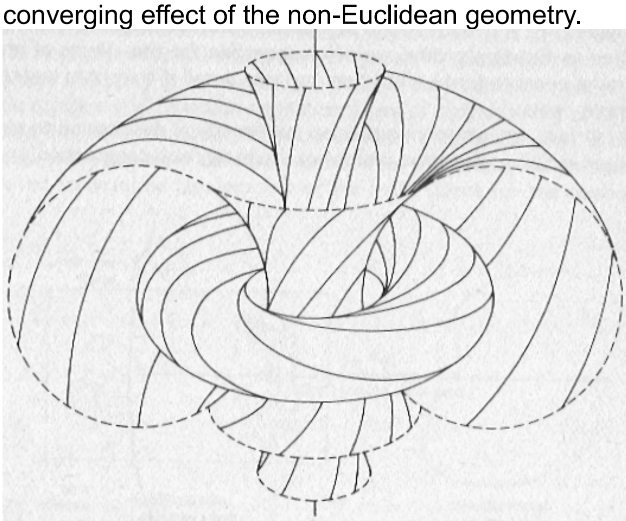


Figure 13. Clifford parallels filling the whole of S^3.

It is possible to fill up the whole of S^3 (or P^3) with a system of lines (great circles on S^3), all of which are parallel to one another in this sense.

Irregular Geometries

The regular geometric models can, at best, represent only the very large scale averaged out spacial geometry of the universe. Local irregularities will occur an a much smaller scale corresponding to the presence of gravitational fields. In Einstein's view these local deviations from Euclidean geometry are totally responsible for the effects of gravity.

The concept of a manifold is required to generalise the concept of a curve or surface which consists of a set of “points” together with some “local structure”. A configuration space in a manifolds is a set of parameters required to completely locate a feature of interest in the manifold (i.e. A 3D object in 3D space has a 6 dimensional configuration space: 3 x position + 3 x orientation)

The most primitive local structure is topology (which represents continuity but not smoothness, distance or size). A cube has the same topology as a sphere (S^2). A sphere with 2 points removed has the same topology as a cylinder.

A topological space is called an n-dimensional topological manifold if it the neighbourhood in the neighbourhood of any point is the same as an n-dimensional euclidean space.
Insert fig 18

A Hausdorff manifold contains no leafs or branches (as a result of being pieced together)

If, in addition, the structure is also differentiable then it will support the ideas of calculus.

Introducing coordinated systems often requires multiple overlapping coordinate systems tied together with transformations but it can become complicated in irregular geometries. However, it is possible and underpins classical differential calculus. Elie Cartan introduced a coordinate-free approach which is based on scalar fields (smooth functions) over-layed by vector fields, covector fields, tensor fields and connections introduced in turn. In this approach, coordinates are an option but not required.

A vector field V is interpreted as a differential operator , which is a function which acts on a scalar field ϕ to produce another scalar field V(ϕ), where the value of V(ϕ) as a point p is the rate of increase of ϕ at p in the direction of the arrow at p.

The set of possible vectors at a particular point forms a tangent space T on manifold M at point p. This arrangement can be “expanded” to form a local “flat” Cartesian coordinate system (despite the manifold M being curved). However, it not an Euclidean space because “distance” is not invariant under transformations within the manifold M.

A covector can be thought of as a linear function which takes a vector field V to a scalar field A(V). It can be visualised as a field of plane elements on M (n-1 dimension of them) each containing all of the directions for which A(V)=0.

A tensor field is a similar map (a multi-linear map) from collections of vectors and covectors to scalars. An important type of tensor field is the Riemannian manifold because it gives an Euclidean structure to a tensor field.

In summary ... “First there is the general concept of a manifold *M,* with its topology and "smoothness" structure. This is the place where the objects of absolute differential calculus can live. But at that stage no specific concept of distance is singled out, even in the tangent spaces. If we wish to introduce such a distance concept, we must do so by specifying a particular metric tensor field g. This enriches the structure of *M* and turns it into what is called a *Riemannian manifold.* In geometrical terms, the assignment of such a Riemannian structure means the assignment of a particular concept of *distance* between any pair of infinitesimally separated points *p, p'.* Hence, stringing such points together into a (smooth) curve on *M* we obtain a concept of distance measured along the curve, called the (Riemannian) *length* of the curve.” pg 104
The requirement for the alternate conformal representation remains.

Euclidean space E^2 is itself a particular example of a Riemannian manifold (as are L^n, S^n, P^n and other irregulars). The deviations from Euclidean geometry can be described on a local scale by the curvature tensor R. The covariant differentiation operation ▼ caries a tensor field ϕ into another slightly more complicated one (denoted ▼ϕ) which measures the rate at which ϕ is varying. An assignment of ▼ to M makes M a manifold with connection.

The geometric meaning of ▼ relates to parallel propagation. However, this notion parallel is different to that in Euclidean space an may involve a rotation when a closed loop in involved. (consider on S^2 travelling from Nth pole, to equator, ¼ way around and then back to the Nth pole). R measures an infinitesimal discrepancy which results from parallel propagation around an infinitesimal closed loop.
Geodesics are the natural analogues for straight lines in M

▼ is usually chosen to be torsion-free (Clifford parallels are not torsion-free)

Einstein's general-relativity is pseudo-Riemannian.

In the space-time of Minkowski $OQ^2=t^2-x^2-y^2-z^2$ "The fact that measurement of time accords with such an expression, rather than with the Newtonian "absolute" time. difference *t - t',* is the key to special relativity. Time is the "distance" measure of Minkowskian geometry. And like ordinary distance in Euclidian geometry, the time measure along a timelike curve (world-line) becomes a *path-dependent* concept. " pg 112 The straight (unacclerated) world-line has maximum length (time interval).

In general-relativity, the flat Minkowski space time M4 is replaced by a curved manifold M. The light cones (null cones) become infinity for particles with mass. Length measures he tim experienced by a particle with a particular curve as its world-line.

The null cones as various points in M importantly define the casual relationship between space-time points.

Black holes have an event horizon around a space-time singularity where curvature goes to infinity.

Freely falling objects have no way to detect the "gravitational force". Well not quite. Geodesic deviation recognises that distant particle in the same physical body feel the "force of gravity" in a slightly different direction (i.e. tidal effects).

The Jacobi equation relates relative acceleration directly to the curvature tensor.

The Friedman models describe a Universe expanding from a singularity and either slowing, stopping or collapsing.

A symplectic manifold is somewhat like that ofa psuedo-Riemannian manifold but where a skew-symmetric tensor takes the place of the symmetric tensor.

Fibre bundles are another type of manifold introduced to manage problems wih differential geometry.

Complex manifolds are very interesting where holomorphic functions are the complex smooth functions. The simplest complex manifold is the complex analogue of a real closed loop – it turns out to be S^2 – a closed complex curve! Complex self transformations on this complex curve relate precisely to the Lorentz group! And this all has application in physics.

Thoughts

Freely falling objects have no way to detect the "gravitational force". Does this relate to how a causal decoupling works? The elements are involved in a "larger" effect but have no way of knowing it – or causing it as such?

The Mathematics of Meteorology
Philip Thompson

Meteorology deals with the structure and behaviour of the atmosphere up about 100km.

Newton, Euler and Bernoulli only has access to four equations of motion relating velocity, density, pressure, Coriolis and gravity

Robert Boyle 1662 discovered linear proportionality between density and pressure.

Jacques Charles 1802 found that pressure was linearly proportional to temperature.

The First Law of Thermodynamics was established at a similar time

By 1844 all of the basic dynamical and thermodynamic laws that govern the behaviour of a fluid like the atmosphere were widely known.

In 1858 Helmholtz discovered some general properties of the solutions to these equations and these were supplemented by work by Kelvin and Rayleigh

Vilhelm Bjerknes 1904 realised that the hydrodynamic equations could be solved in their general form and that the future of the weather depended on the past.

Richardson attempted a set of solutions by hand during WWI on a large grid layout over Europe and obtained a result similar to sound waves – a glorious failure. He wistfully imagined enough computation power in the dim future – 25 years later.

Due to the difficulties involved with the non-linear aspects of the equations there was considerable work done with linearised versions which never-the-less advanced the wave motion behaviours considerably.

A critical step was Neumann's separation of program from data in mechanical calculations.

Charney (1948) suggested that a derived form of the hydrodynamic equations could be modified in such a way that solutions corresponding to high-speed sound and gravity waves (both of which lead to computational instability) are excluded, but such that solutions corresponding to the large-scale “meteorological” modes are retained almost intact. It turned out that exactly those features of the atmospheric motions made the general problem difficult.

“The question of filtering is a ... subtle one. With regard to the atmosphere's large-scale behaviour, the existence of gravitational forces is very important from some points of view but relatively unimportant from others. In this case, the problem of discarding certain features of the system while retaining the meteorological essence is not so simple as merely omitting terms from one of many possible formulations of the general system of equations, It might be expected that similar questions would arise in the analysis of any system which is capable of displaying widely different modes of behaviour under different external conditions or which, under normal conditions, displays a dominant mode of behaviour. In such cases, the methods of filtering developed by the meteorologist may suggest systematic approaches to problem simplification in other fields. In this respect, meteorology and weather forecasting, through their indigenous methods of multiple scale analysis, have undoubtedly contributed to applied mathematics and the general area of continuum mechanics. “ pg 144

Weather prediction was prototyped in 1950 and into production in 1954.

The filtering method was attacked, the conditions under which a natural state of balance between pressure and velocity waves reduced the amplitude of spurious sound and gravity waves was established – but it left the Rossby-wave equations essentially intact (and essentially still current)

There are definite limits to the finite difference method in term of both initial conditions and memory/processing load.

Fast Fourier Transforms were successfully developed and utilised

Lagrangian methods combine exact analytical methods with a bare minimum of approximate numerical methods with the advantage that the size and shape of grid elements can be chosen arbitrarily and variably.

Current work is exploring the “transfer of uncertainty” from unresolvably small scales to large scales.

Further Reading

“multiple scale analysis” and continuum mechanics”

The Four Color Problem
Kenneth Appel and Wolfgang Haken

In 1976 the Four-Color problems was solved.

In 1852 Francis Guthrie wrote to his brother Frederick observing that the countries on every map appeared to be able to be colored with just four colors and asked if there was a proof.

Combinatorial Scheduling Theory
Ronald Graham

Scheduling problems occur very frequently.

The basic scheduling problem is to determine how the finishing time depends on the processing times and the precedence constraints of the tasks, the number of processors, and the strategies used for constructing the schedules. In particular, we would like a method to determine schedules that have the earliest possible finishing time.

In a worst case analysis $\frac{f'}{f} \leq 2 - \frac{1}{m}$ where f is the finishing time and m is the number of processors.

Critical path scheduling forms the basis of developing PERT charts where tasks that head the current critical paths are chosen as the next tasks to execute. However, this approach has no guarantee of an optimum schedule and can result in the near worst!

When Critical Path scheduling is used for independent tasks $\frac{f_{CP}}{f_{OPT}} = \frac{4}{3} - \frac{1}{3\mathrm{m}}$ where f is the finishing time (CriticalPath and OPTimum) and m is the number of processors.

Scheduling problems are in general NP-complete

There are some know methods which can generate near optimal solutions in reasonable computational time.

There are some special cases where optimum solutions can be found: 1. where all tasks take 1 time unit and precedence relations form a tree structure; 2. where all tasks take 1 time unit and there are only 2 processors (the FKN algorithm, the CG algorithm and the GJ algorithm).

Bin packing is the inverted question of “how many processors are required to complete the task by a given deadline. Bin packing problems are often extremely difficult.

First-fit packing fills each bin to the maximum with items of size L without overflow before proceeding to the next. $FF(L) \leq \frac{17}{10} OPT(L) + 2$ The factor 17/10 comes from so called Egyptian fractions where fractions are expressed as a sum of fractions with numerator 1 (i.e. 25/28 = ½ + ¼ + 1/7)

First-fit decreasing packing sorts the sizes into decreasing order prior to completing same procedure and results in a superior result.

$$FFD(L) \leq \frac{11}{9} OPT(L) + 4$$

The 2D bin packing problem is also interesting and difficult with some counter intuitive results.

Statistical Analysis of Experimental Data
David S. Moore

Statistics involves collecting data, organising and summarising data, and drawing conclusions from the data (statistical inference)

This article focusses only on inference from statistical data and concentrates on only one approach; random paths.

Data collection must be done in such a way that it can be meaningfully interpreted. Comparative studies must avoid sampling bias.

Classical studies assumed that the number of samples were chosen and allocated in advance. However, sometimes the samples arrive sequentially (medical trials) rather than as a batch.

It is invalid to simply stop collecting data when the answer is favourable because a significance test of 5% will be exceeded 1 in 20 times. Over infinite trials, a random walk will cross any significance boundary infinite times.

Khintchine (1924) discovered the law of the iterated logarithm which describes how widely a random walk will fluctuate. The boundaries lie $\sqrt{2 \log \log n}$ standard deviations on either side of the mean.

In sequential analysis the data is used to decide when to stop collecting data <u>and</u> to draw inferences. The sequential probability ratio test (Wald) continues sampling until one of two parallel boundaries are crossed. If the lower boundary is crossed then the we stop and conclude that the effect is random. If the upper boundary is crossed then we stop and conclude that the effect is real. This is both simple and optimal. SPRT on average requires less observations to reach a decision than any other procedure having the same or smaller probability of error.

SPRT has been extended to compare two alternatives, to specify a maximum number of trials (when they are expensive), to operate in batch mode, and for multiple options (rather than just yes-no decisions)

Zelen (1969) introduced a introduced the play-the-winner rule which assigns sequential patients to a treatment so long as the treatment succeeds and switch treatments when a failure occurs. If the result is not available immediately then a success generates a future trial and each patient is assigned out of the “hat”.

A biased coin assignment assigns each sequential treatment with probability ½ if there are an equal number of patient receiving each treatment and 2/3 if there is a lesser number of patients receiving the treatment. A more practical approach is to assign treatments randomly in pairs.

The Darling-Robins procedure enables inference without error by analysing that rate at which the random walk converges but it has the price that sampling may proceed for ever (there is no guaranteed stop)

The optimality of SPRT establishes it as a standard and point of comparison for alternate procedures designed with alternate definitions of optimality.

The secretary problem involves hiring a secretary with the N candidates presented in random order but the employer must either hire the candidate at the end of the interview or call for another interview. The solution is obtained with the backwards induction procedure which at each point considers the probability of obtaining a better candidate in those remaining. The optimum is to pass about 37% of the candidates and choose the next best.

What is a Computation?
Martin Davis

Turing defined the notion of a computation which led to a proof that some problems are unsolvable.

Turing, Post and others defined computation in a purely mechanical fashion. Turing defined the Turin machine based on a process of coding the problem into a set of symbols followed by a mechanical symbol manipulation. He envisaged symbols on an infinite tape and a machine working on one symbol at a time and halting only when the computation was complete.

The ***Universal Turing Machine*** was capable of simulating the operation of any Turing machine.

The ***halting problem*** is that there is no computational procedure for testing a give tape expression to determine whether or not the universal program U will eventually halt when begun with that tape configuration.

Hilbert wanted a computational procedure for testing whether or not the conclusion can be deduced using the rules of logic from the premise. However this is impossible due to the halting problem.

Thue showed that the halting problem also leads to an unsolvable word problem. Word problems begin with an alphabet of symbols and any string of letters is a word. A word problem is specified by writing down a finite list of equations between words. Other equations can be derived by substitution. The problem is to determine if one word is equivalent to another by transformation. Word problems have had significant application in group theory and topology.

Diophantine equations seek integer solutions to a set of equations. Hilbert's tenth problem sought a mechanical procedure for testing to see if Diophantine a set of equations have a solution. Matiyasevich showed that this problem is insolvable using surprisingly elementary mathematics.

Bertrand Russell and Alfred Whitehead showed that all existing mathematical proofs could be translated into the logic system detailed in principa mathematica. Godel showed that there were statements about whole number which could be neither proved or disproved – **undecidable**. This result can be echoed in a Turing machine context. Godel's theorem asserts that no rules of proof can be both sound and complete.

More than 99.8% of all strings of length n have complexity > n-10. There is some definite number k_0, such that it is in principle impossible, by ordinary

Mathematical methods, to prove that any string of bits has complexity greater than k_0.

Mathematics as a Tool for Economic Understanding
Iacob Schwartz

The practical value of mathematics over common sense is that:

1. mathematical arguments remain sound even if they are long and complex,
2. axioms stake out an area within which patterns of reasoning have a reproducible, objective character, and
3. mathematics allows an immense variety of intellectual structures to be elaborated.

We only have mathematics and empirical science to check the validity of our thinking. Compelling models are required to overcomes mathematics stark formality. Common sense is based on intuitive appeal and plausibility and Mathematics finds its strength when neither of these are true.

Economics theory concerns the mechanisms by which society pushes its members to produce activity and allocates the fruits of that activity. American society emphasises the importance of unhampered individual production and choice. Theory is required to guide and support government intervention that are in apparent conflict with these ideals.

The Newman-Morgenstern theory of games assumes a finite number of participants, whose interactions are constrained, but not uniquely determined, by a set of precisely defined pre-specified rules. The rules need not be symmetric between participants. A set of rewards are distributed to each player after each move. The participants are driven to interact because the reward to a particular participant is determined jointly by both their actions and those of one or more other participants.

Four instructive games are:

Bluff is a two person zero-sum game with payoffs: truth accepted = -10; truth questioned +50; bluff successful = +50; bluff foiled = -100.
Cooperation where two people can choose to cooperate or not with pay-offs for the two players: one selfish player defects (+2/-1); both players defect (-1/-1); cooperate B (-0.9/-0.9)
Litter is a multi-player game where there is a small cost to place litter in a bin, a large but unlikely fine for littering, and a shared cost proportional to the total amount of littering.
Stagnation is a multi-player game where if more than half choose to 'go' (to a social engagement) then the payoff to those that 'go' = $10. If less than half 'go' then the payoff is $-1.

Games such as paper-scissors-rock do not have a stationary best defensive strategy. However, Newman realised that this changes if players are allowed to randomise their moves but with fixed probability. The 'value' of a two-person zero-sum game is the amount won by one player if both players use their best defensive strategies – it is a measure of the fairness of the game.

Newman and Morgenstern then proceeded to analyse the stability of coalitions beginning with 3 player games for which it is best to choose an ally and divide the rewards equally. As the number of rewards increase the complexity of the analysis increases rapidly and there is a rich variety of phenomenon. However, this is a poor basis for economics due to the complexity and the assumption that players will have the time and resources to reach agreements completely outside the rules of the game that are enforceable.

John Nash introduced the concept of coalition free equilibrium for which the validity increases as the number of participants increase. An equilibrium is reached when no individual player can move to a more advantageous strategy. This can be a sub-optimal equilibrium if cooperation would result in a better outcome for every player. Often sub-optimal equilibrium deliver very near to a minimal pay-off to the players which forms a strong argument for centralised economic intervention. And there is such a thing as a free lunch!

Some key elements in economic modelling include: the material profitability of production; the pattern of distribution (of profit); the pattern of consumption (vs. savings); the profit maximisation strategy of productive enterprises; factors relating to the housing and mortgage market; the stable spending patterns of retired persons and government.

The influential Wharton econometric model and others are able to provide reliable forecasts for about a year limited by inaccuracies and non-linearities. This is ample time for adjustment.

Boom/bust cycles are related to the gradual accumulation of excess inventories during boom periods and the manner in which rising interest rates of such periods act to choke off housing sales. Economic policy makers have attempted to move the economy to higher levels of product by raising demand (personal, investment, government) through lower personal income tax, increasing pensions or the number of pensioners, or the size of the armed forces, encouraging investment through rebates, or direct public infrastructure investments.

There is a risk of intervention of other suboptimal equilibria. To implement these measures the government requires rewards in massive amounts. In practice this is usually notes of government indebtedness which constitute national debt. Hyper-inflation can be avoided with property tax. Inflation is a price related sub-optimality. Prices follow wages (with no equilibria) and unemployment (highly undesirable) limits wage rises.

Mathematical Aspects of Population Biology
Frank C. Hoppensteadt

Population biology is primarily concerned with counting, estimating and predicting population sizes. It has application in bacterial sewerage treatment, natural resource management, geographic distribution of genes, age distribution of populations, and the spread of disease in forests.

In some cases a population may behave in a chaotic fashion despite being governed by simple biological law.

The periodic emergence of seventeen-year locusts (cicadas) can be modelled based on a limited environmental carrying capacity and a responding predator threshold. The balance of these two parameters is rather finely tuned between emergence every year and extinction. Syncronised emergence only occurs with life spans in excess of 10 years. It is an example of a non-linear oscillator.

The Beverton-Holt model can describe how the dynamics of a fish population can be determined by the cobwebbing method once the reproduction curve is known.

However, it is easy to identify reproduction curves which result in chaotic population changes.

Chaotic systems are typical in many systems and generate random outcomes. It is surprising that there is any order in nature!

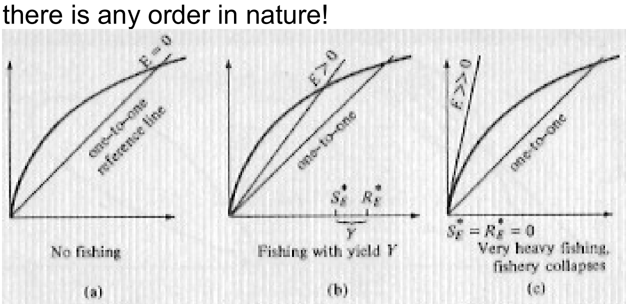


Figure 10. Three examples of yield when various efforts are exerted.

The Beverton-Holt model can be used to estimate the impact of a relative increase in fishing effort. However the actual reproduction curve can vary widely and even a small variation in the curve from that estimated can have a catastrophic result.

An additional problem is that this only considers biological yield and not economic yield. Often it is economically beneficial to harvest the resource to extinction because the cash return is far higher than the natural resource return at sustainable rates.

The spread of disease can be modelled as CA.

The age structure of a human population can be predicted using renewal theory based on survival probabilities and fertilities.

In the absence of outside influence a genes frequency will not change from one generation to the next (the Hardy-Weinberg equilibrium)

Blood groups have been a powerful tool in anthropology.

The Relevance of Mathematics
Felix E. Browder and Saunders Mac Lane

This chapter contains a broad magical mystery tour of mathematics with the following conclusions.

The potential usefulness of a mathematical concept or technique in helping to advance scientific understanding has very little to do with what one can foresee before that concept or technique has appeared.

Such usefulness has very little to do with the purity or applied character of the motivation underlying the creation of the technique or concept, or with its degree of abstraction.

Concepts or techniques are only useful if they can be eventually put in a form which is simple and relatively easy to use in a variety of contexts.

We don't know what will be useful (or even essential) until we have used it. We can't rely upon the concepts and techniques which have been applied in the past, unless we want to rule out the possibility of significant innovation.

To summarize, mathematics as the science of significant form interacts in an ever widening way with the whole framework of human thought and practice.

It was a thought of this sort that was expressed by the Anglo-American philosopher Alfred North Whitehead when he put forward his own rewriting of Plato's Lecture on the Good in an intellectual credo entitled "Mathematics and the Good" written for the volume dedicated to him in the Library of Living Philosophers. Whitehead wrote: 'The notion of the importance of pattern is as old as civilization. Every art is founded on the study of pattern. The cohesion of social systems depends on the maintenance of patterns of behaviour, and advances in civilization depend on the fortunate modification of such behaviour patterns. Thus the infusion of patterns into natural occurrences and the stability of such patterns, and the modification of such patterns is the necessary condition for the realization of the Good. Mathematics is the most powerful technique for the understanding of pattern, and for the analysis of the relation of patterns. Here we reach the fundamental justification for the topic of Plato's lecture. Having regard to the immensity of its subject matter, mathematics, even modern mathematics, is a science in its babyhood. If civilization continues to advance in the next two thousand years, the overwhelming novelty in human thought will be the dominance of mathematical understanding."